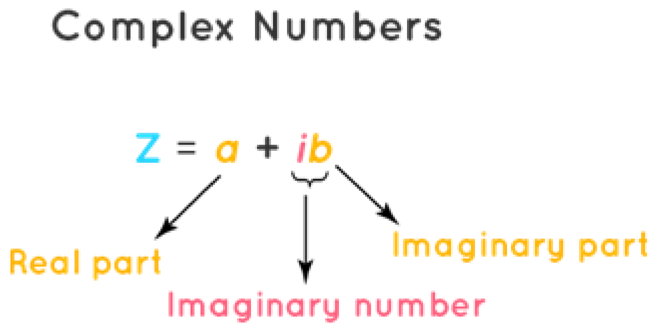
Electronic Circuits

Lecture 1.3: Mathematical Background

Complex Number Representations

Representation of a Complex Number



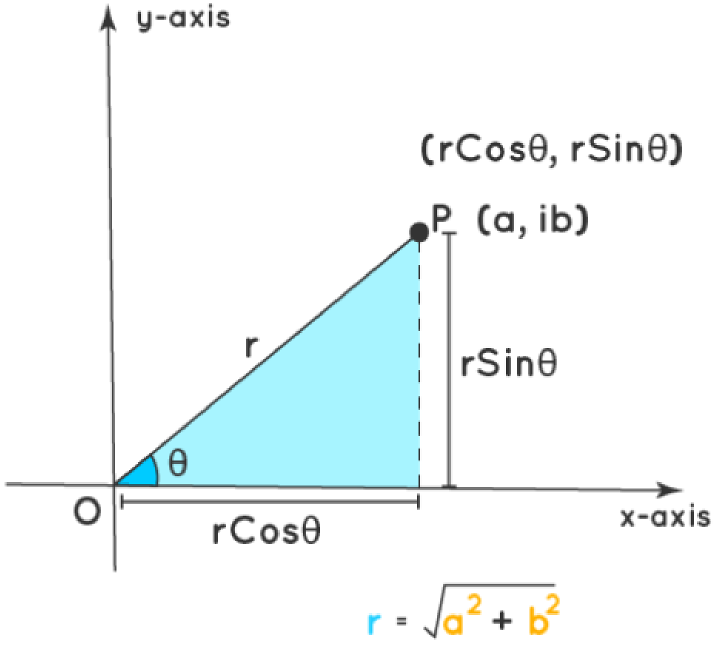
The conjugate of a complex number $z = a + ib$ is

$\bar{z} = a - ib$.

The magnitude of a complex number $z = a + ib$ is

$|z| = \sqrt{a^2 + b^2}$.

Representation of a Complex Number



$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = i.i^2 = i(-1) = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^{4n} = 1$

$i^{4n+1} = i$

$i^{4n+2} = -1$

$i^{4n+3} = -i$

Euler Formula

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \cdots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^{ix} = \cos x + i \sin x$$

Laplace Transform Tables (1)

$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\,dt$
$f + g$	$F + G$
$\alpha f \ (\alpha \in \mathbf{R})$	αF
$\dfrac{df}{dt}$	$sF(s) - f(0)$
$\dfrac{d^k f}{dt^k}$	$s^k F(s) - s^{k-1} f(0) - s^{k-2}\dfrac{df}{dt}(0) - \cdots - \dfrac{d^{k-1} f}{dt^{k-1}}(0)$
$g(t) = \int_0^t f(\tau)\,d\tau$	$G(s) = \dfrac{F(s)}{s}$
$f(\alpha t),\ \alpha > 0$	$\dfrac{1}{\alpha} F(s/\alpha)$
$e^{at} f(t)$	$F(s-a)$
$tf(t)$	$-\dfrac{dF}{ds}$
$t^k f(t)$	$(-1)^k \dfrac{d^k F(s)}{ds^k}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
$g(t) = \begin{cases} 0 & 0 \le t < T \\ f(t-T) & t \ge T \end{cases},\ T \ge 0$	$G(s) = e^{-sT} F(s)$

Laplace Transform Tables (2)

1	$\frac{1}{s}$
δ	1
$\delta^{(k)}$	s^k
t	$\frac{1}{s^2}$
$\frac{t^k}{k!},\ k \geq 0$	$\frac{1}{s^{k+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2} = \frac{1/2}{s-j\omega} + \frac{1/2}{s+j\omega}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2} = \frac{1/2j}{s-j\omega} - \frac{1/2j}{s+j\omega}$
$\cos(\omega t + \phi)$	$\frac{s\cos\phi - \omega \sin\phi}{s^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$

Laplace Transform Examples (1)

$$f(t) = 6\mathbf{e}^{-5t} + \mathbf{e}^{3t} + 5t^3 - 9$$

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

$$g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$$

$$G(s) = 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

Laplace Transform Examples (2)

$$h(t) = 3\sinh(2t) + 3\sin(2t)$$

$$H(s) = 3\frac{2}{s^2 - (2)^2} + 3\frac{2}{s^2 + (2)^2}$$

$$= \frac{6}{s^2 - 4} + \frac{6}{s^2 + 4}$$

$$g(t) = \mathbf{e}^{3t} + \cos(6t) - \mathbf{e}^{3t}\cos(6t)$$

$$G(s) = \frac{1}{s-3} + \frac{s}{s^2 + (6)^2} - \frac{s-3}{(s-3)^2 + (6)^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2 + 36} - \frac{s-3}{(s-3)^2 + 36}$$

Inverse Laplace Transform Examples (1)

$$F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$$

$$= 6\,\frac{1}{s} - \frac{1}{s-8} + 4\frac{1}{s-3}$$

$$f(t) = 6\,(1) - \mathbf{e}^{8t} + 4\left(\mathbf{e}^{3t}\right)$$

$$= 6 - \mathbf{e}^{8t} + 4\mathbf{e}^{3t}$$

$$H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$$

$$= \frac{19}{s-(-2)} - \frac{1}{3\left(s-\frac{5}{3}\right)} + \frac{7\frac{4!}{4!}}{s^{4+1}}$$

$$= 19\frac{1}{s-(-2)} - \frac{1}{3}\,\frac{1}{s-\frac{5}{3}} + \frac{7}{4!}\,\frac{4!}{s^{4+1}}$$

$$h(t) = 19\mathbf{e}^{-2t} - \frac{1}{3}\mathbf{e}^{\frac{5t}{3}} + \frac{7}{24}t^4$$

Inverse Laplace Transform Examples (2)

$$F(s) = \frac{6s}{s^2+25} + \frac{3}{s^2+25}$$

$$= 6\frac{s}{s^2+(5)^2} + \frac{3\frac{5}{5}}{s^2+(5)^2}$$

$$= 6\frac{s}{s^2+(5)^2} + \frac{3}{5}\frac{5}{s^2+(5)^2}$$

$$f(t) = 6\cos(5t) + \frac{3}{5}\sin(5t)$$

$$G(s) = \frac{8}{3s^2+12} + \frac{3}{s^2-49}$$

$$= \frac{1}{3}\frac{8}{s^2+4} + \frac{3}{s^2-49}$$

$$= \frac{1}{3}\frac{(4)(2)}{s^2+(2)^2} + \frac{3\frac{7}{7}}{s^2-(7)^2}$$

$$g(t) = \frac{4}{3}\sin(2t) + \frac{3}{7}\sinh(7t)$$

Inverse Laplace Transform Examples (3)

$$F(s) = \frac{1-3s}{s^2+8s+21}$$

$$\begin{aligned} s^2+8s+21 &= s^2+8s+16-16+21 \\ &= s^2+8s+16+5 \\ &= (s+4)^2+5 \end{aligned}$$

$$F(s) = \frac{1-3s}{(s+4)^2+5}$$

$$\begin{aligned} F(s) &= \frac{1-3(s+4-4)}{(s+4)^2+5} \\ &= \frac{1-3(s+4)+12}{(s+4)^2+5} \\ &= \frac{-3(s+4)+13}{(s+4)^2+5} \end{aligned}$$

$$F(s) = -3\frac{s+4}{(s+4)^2+5} + \frac{13\frac{\sqrt{5}}{\sqrt{5}}}{(s+4)^2+5}$$

$$f(t) = -3\mathbf{e}^{-4t}\cos\left(\sqrt{5}t\right) + \frac{13}{\sqrt{5}}\mathbf{e}^{-4t}\sin\left(\sqrt{5}t\right)$$

Solving Linear Equations (1)

Addition and Subtraction Properties of Equality:

Let a, b, and c represent algebraic expressions.

1. **Addition property of equality:**

   If $a = b$,

   then $a + c = b + c$

2. **Subtraction property of equality:**

   If $a = b$,

   then $a - c = b - c$

Multiplication and Division Properties of Equality:

Let a, b, and c represent algebraic expressions.

1. **Multiplication property of equality:**

   If $a = b$,

   then $ac = bc$

2. **Division property of equality:**

   If $a = b$,

   then $\frac{a}{c} = \frac{b}{c}$ (provided $c \neq 0$)

Solving Linear Equations (2)

Clearing Fractions or Decimals in an Equation:

When solving an equation with fractions or decimals, there is an option of clearing the fractions or decimals in order to create a simpler equation involving whole numbers.

1. To clear fractions, multiply both sides of the equation (distributing to all terms) by the LCD of all the fractions.
2. To clear decimals, multiply both sides of the equation (distributing to all terms) by the lowest power of 10 that will make *all* decimals whole numbers.

Steps for Solving a Linear Equation in One Variable:

1. Simplify both sides of the equation.
2. Use the addition or subtraction properties of equality to collect the variable terms on one side of the equation and the constant terms on the other.
3. Use the multiplication or division properties of equality to make the coefficient of the variable term equal to 1.
4. Check your answer by substituting your solution into the original equation.

Solving Linear Equations (3): Example 1

$$10 - 5x = 3(x - 4) - 2(x + 7)$$
$$10 - 5x = 3x - 12 - 2x - 14$$
$$10 - 5x = x - 26$$
$$\underline{\quad -x \quad -x \quad}$$
$$10 - 6x = -26$$
$$\underline{-10 \qquad\quad -10}$$
$$-6x = -36$$
$$\frac{-6x}{-6} = \frac{-36}{-6}$$
$$x = 6$$

Check:

$$10 - 5(6) = 3(6 - 4) - 2(6 + 7)$$
$$10 - 30 = 3(2) - 2(13)$$
$$-20 = 6 - 26$$

$-20 = -20$ (Solution Checks)

Solving Linear Equations (4): Examples 2 and 3

$\frac{x-2}{5} - \frac{x-4}{2} = 2$ (LCD is 10)

$$10\left(\frac{x-2}{5} - \frac{x-4}{2}\right) = 2(10)$$

$$\frac{10}{1} \cdot \frac{(x-2)}{5} - \frac{10}{1} \cdot \frac{(x-4)}{2} = 20$$

$$2(x-2) - 5(x-4) = 20$$

$$2x - 4 - 5x + 20 = 20$$

$$-3x + 16 = 20$$

$$\underline{\quad -16 \quad -16}$$

$$\frac{-3x}{-3} = \frac{4}{-3}$$

$$x = -\frac{4}{3}$$

$$0.05x + 0.25 = 0.2$$

$$100(0.05x + 0.25) = 0.2(100)$$

$$5x + 25 = 20$$

$$\underline{\quad -25 \quad -25}$$

$$\frac{5x}{5} = \frac{-5}{5}$$

$$x = -1$$

Check:

$$0.05(-1) + 0.25 = 0.2$$

$$-0.05 + 0.25 = 0.2$$

$0.2 = 0.2$ (Solution Checks)

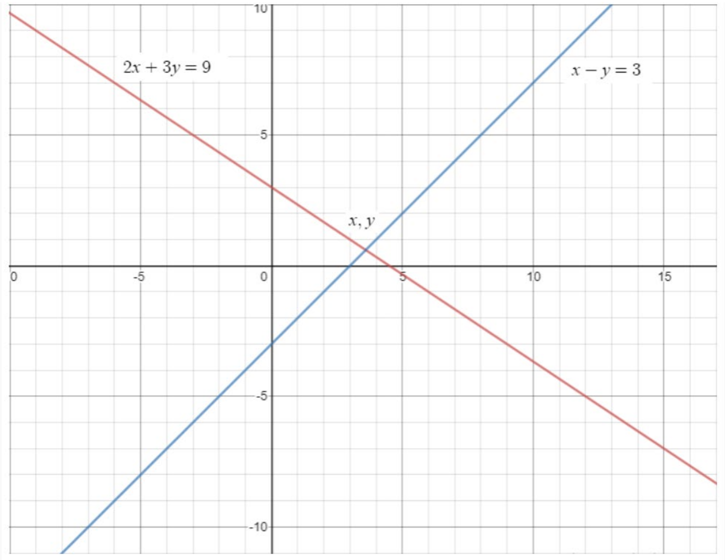
Solving Linear Equation Systems (1)

- Graphical Method
- Elimination Method
- Substitution Method
- Cross Multiplication Method
- Matrix Method
- Determinants Method

Solving Linear Equation Systems (2)

Graphical Method

The graph of 2x + 3y = 9 and x − y = 3 will be as follows:



Elimination Method

$$
\begin{aligned}
&2x + 3y = 9 \\
&(x - y = 3) * (-2)
\end{aligned}
$$

$$
\begin{aligned}
&2x + 3y = 9 \\
+\ &-2x + 2y = -6
\end{aligned}
$$

$5y = 3$ --> $y=3/5$

$2x+3(3/5)=9$ --> $2x = 9-9/5$ --> $2x=36/5$ --> $x=36/10$

Solving Linear Equation Systems (3)

Substitution Method

$2x + 3y = 9$
$x - y = 3 \dashrightarrow x = y + 3$

$2(y+3) + 3y = 9 \dashrightarrow 5y + 6 = 9 \dashrightarrow 5y = 3 \dashrightarrow y = 3/5$

$x = 3/5 + 3 = 18/5 = 36/10$

Cross-Multiplication Method*

$2x + 3y = 9$
$x - y = 3$

$a1 = 2, b1 = 3, c1 = -9$
$a2 = 1, b2 = -1, c2 = -3$

For 2-variable case

$x = (b1c2-b2c1)/(b2a1-b1a2)$
$y = (c1a2-c2a1)/(b2a1-b1a2)$

$x = 18/5 = 36/10$
$y = 3/5$

*do not prefer, require memorization more

Solving Linear Equation Systems (4)

Matrix Method

$$2x + 3y = 9$$
$$x - y = 3$$

$$\begin{bmatrix} 2x + 3y \\ 1x + -1y \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

Matlab Code

```
A = [2 3; 1 -1];
b = [9 ; 3];
sol = pinv(A)*b
sol = A\b
```

Determinants Method*

$x = \Delta_1/\Delta,$

$y = \Delta_2/\Delta$

For 2-variable case

$$\Delta_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \ \Delta_2 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \ and \ \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

*do not prefer, require memorization more similar to Cross-Multiplication Method

Solving Linear Equation Systems (5)

To solve a linear equation with fraction, follow these steps:

- Step 1: Make any complex fraction into a simple fraction
- Step 2: Find the LCM of all denominators
- Step 3: Multiply the equation with the LCM of the denominator
- Step 4: Cancel out the fractions as all the denominators can be divided by the LCM value
- Step 5: Solve the final linear equation using any of the methods explained here

Solving Linear Equation Systems (6)

Find a solution to the following system:

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y - z &= -6 \\ 2x - 5y + 5z &= 17 \end{aligned} \qquad (1, -1, 2)$$

Find a solution to the following system:

$$\begin{aligned} 2x + y - 2z &= -1 \\ 3x - 3y - z &= 5 \\ x - 2y + 3z &= 6 \end{aligned} \qquad (-1, 1, 2)$$

Thanks for listening ☺

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