Binary Systems

BME208 – Logic Circuits
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Binary Numbers 1/2

- Internally, information in digital systems is of binary form
  - groups of bits (i.e. binary numbers)
  - all the processing (arithmetic, logical, etc) are performed on binary numbers.
- Example: 4392
  - In decimal, 4392 = …
  - Convention: write only the coefficients.
  - $A = a_1 a_0 . a_{-1} a_{-2} a_{-3}$ where $a_j \in \{0, 1, \ldots, 9\}$
  - How do you calculate the value of A?

Binary Numbers 2/2

- Decimal system
  - coefficients are from {0,1, …, 9}
  - and coefficients are multiplied by powers of 10
  - base-10 or radix-10 number system
- Using the analogy, binary system {0,1}
  - base(radix)-2
- Example: 25.625
  - 25.625 = decimal expansion
  - 25.625 = binary expansion
  - 25.625 =

Base-r Systems

- base-r (n, m)
  - $A = a_{n-1} r^{n-1} + \ldots + a_1 r^1 + a_0 r^0 + a_{-1} r^{-1} + a_{-2} r^{-2} + \ldots + a_{-m} r^{-m}$
- Octal
  - base-8 = base-2^3
  - digits {0,1, …, 7}
  - Example: $(31.5)_8$ = octal expansion =
- Hexadecimal
  - base-16
  - digits {0, 1, …, 9, A, B, C, D, E, F}
  - Example:
  - $(19.A)_{16}$ = hexadecimal expansion =

Powers of 2

- 2^{10} = 1,024 (K) -
- 2^{20} = 1,048,576 (M) -
- 2^{30} → (G) -
- 2^{40} → (T) -
- 2^{50} → (P) -
- exa, zetta, yotta, ... (exbi, zebi, yobi, ...)
- <u>Examples</u>:
  - A byte is 8-bit, i.e. 1 B
  - 16 GB = ? B = 17,179,869,184

Arithmetic with Binary Numbers

	10101	21	augend
+	10011	19	addend
1	01000	40	sum

	10101	21	minuend
-	10011	19	subtrahend
0	00010	2	difference

				0	0	1	0	multiplicand (2)
			×	1	0	1	1	multiplier (11)
				0	0	1	0	
			0	0	1	0		
		0	0	0	0			
+	0	0	1	0				
	0	0	1	0	1	1	0	product (22)

Multiplication with Octal Numbers

				3	4	5	229	multiplicand
			×	6	2	1	401	multiplier
				3	4	5		
			7	1	2			
+	2	5	3	6				
	2	6	3	2	6	5	91829	product

Base Conversions

- From base-r to decimal is easy
  - expand the number in power series and add all the terms
- Reverse operation requires division
- Simple idea:
  - divide the decimal number successively by r
  - accumulate the remainders
- If there is a fraction, then integer part and fraction part are handled separately.

Base Conversion Examples 1/3

- <u>Example 1</u>:
  - 55
  - (decimal to binary)

55	1 ↑	1
27	1	2
13	1	4
6	0	
3	1	16
1=	1	32

- <u>Example 2</u>:
  - 144
  - (decimal to octal)

144	0 ↑	$0x8^0$
18	2	$2x8^1$
2=	2	$2x8^2$

Base Conversion Examples 2/3

- Example 1: 0.6875 (decimal to binary)
  - When dealing with fractions, instead of dividing by r multiply by r until we get an integer
  - $0.6875 \times 2 = 1.3750 = 1 + 0.375 \rightarrow a_{-1} = 1$
  - $0.3750 \times 2 = 0.7500 = 0 + 0.750 \rightarrow a_{-2} = 0$
  - $0.7500 \times 2 = 1.5000 = 1 + 0.500 \rightarrow a_{-3} = 1$
  - $0.5000 \times 2 = 1.0000 = 1 + 0.000 \rightarrow a_{-4} = 1$
  - $(0.6875)_{10} = (0.1011)_2$

Base Conversion Examples 2/3

- We are not always this lucky
- Example 2: (144.478) to octal
  - Treat the integer part and fraction part separately
  - $0.478 \times 8 = 3.824 = 3 + 0.824 \rightarrow a_{-1} = 3$
  - $0.824 \times 8 = 6.592 = 6 + 0.592 \rightarrow a_{-2} = 6$
  - $0.592 \times 8 = 4.736 = 4 + 0.736 \rightarrow a_{-3} = 4$
  - $0.736 \times 8 = 5.888 = 5 + 0.888 \rightarrow a_{-4} = 5$
  - $0.888 \times 8 = 7.104 = 7 + 0.104 \rightarrow a_{-5} = 7$
  - $0.104 \times 8 = 0.832 = 0 + 0.832 \rightarrow a_{-6} = 0$
  - $0.832 \times 8 = 6.656 = 6 + 0.656 \rightarrow a_{-7} = 6$
  - $144.478 = (220.3645706\ldots)_8$

Conversions between Binary, Octal and Hexadecimal

- r = 2 (binary), r = 8 (octal), r = 16 (hexadecimal)

```
10110001101001.101100010111

10 110 001 101 001.101 100 010 111

10 1100 0110 1001.1011 0001 0111
```

- Octal and hexadecimal representations are more compact.
- Therefore, we use them in order to communicate with computers directly using their internal representation

Complement

- Complementing is an operation on base-r numbers
- Goal: To simplify subtraction operation
  - Rather turn the subtraction operation into an addition operation
- Two types
  1. Radix complement (r’s complement)
  2. Diminished complement ((r-1)’s complement)
- When r = 2
  1. 2’s complement
  2. 1’s complement

How to Complement?

- A number N in base-r (n-digit)
  1. $r^n - N$ r's complement
  2. $(r^n-1) - N$ (r-1)'s complement
  - – where n is the number of digits we use
- Example: Base r = 2, #Digits n = 4, Given N = 7
  - – $r^n = 2^4 = 16$, $r^n - 1 = 15$.
  - – 2's complement of 7 → 9
  - – 1's complement of 7 → 8
- Easier way to compute 1's and 2's complements
  - – Use binary expansions
  - – 1's complement: negate
  - – 2's complement: negate + increment

How to Complement?

- 10’s complement of 9 is 0+1=1
- 10’s complement of 09 is 90+1=91
- 10’s complement of 009 is 990+1=991
- 9’s complement of 9 is 0
- 9’s complement of 09 is 90
- 9’s complement of 009 is 990
- 2’s complement of 100 is 011+1=100
- 2’s complement of 111 is 000+1=001
- 2’s complement of 000 is 000
- 1’s complement of 11110001 is 00001110

Subtraction with Complements 1/4

- Conventional subtraction
  - Borrow concept
  - If the minuend digit is smaller than the subtrahend digit, you borrow “1” from *a* digit in higher significant position

	minuend
−	subtrahend
	difference

- With complements
  - M-N = ?
  - $r^n – N$ r’s complement of N
  - $M + (r^n – N) =$

Subtraction with Complements 2/4

- $M + (r^n – N) = M – N + r^n$

1. if $M \geq N$,
   – the sum will produce a carry, that can be discarded
2. Otherwise,
   – the sum will not produce a carry, and will be equal to $r^n – (N\text{-}M)$, which is the r’s complement of N-M

Subtraction with Complements 3/4

- Example:
  - X = 101 0100 (84) and Y = 100 0011 (67)
  - X-Y = ? and Y-X = ?

	X	1010100	84
2's complement of	Y	+ 0111101	2's comp of 67
		1 0010001	17
	Y	1000011	67
2's complement of	X	+ 0101100	2's comp of 84
		1101111	111

2's complement of X-Y

Subtraction with Complements 4/4

- Example: Previous example using 1's complement
  - X = 101 0100 (84) and Y = 100 0011 (67)

	X		1010100	84
1's complement of	Y	+	0111100	1s comp of 67
			1 0010000	16

Increase by 1 to get X-Y

	Y		1000011	67
1's complement of	X	+	0101011	1s comp of 84
			1101110	110

1's complement of X-Y

Signed Binary Numbers

- Pencil-and-paper
  - Use symbols “+” and “-”
- We need to represent these symbols using bits
  - Convention:
    - 0 positive
1 negative
    - The leftmost bit position is used as a sign bit
  - In <u>signed representation</u>, bits to the right of sign bit is the number
  - In <u>unsigned representation</u>, the leftmost bit is a part of the number (i.e. the most significant bit (MSB))

Signed Binary Numbers

- Example: 5-bit numbers
  - 01011 → (unsigned binary) Number is 11
  - → (signed binary) Number is +11
  - 11011 → (unsigned binary) Number is 27
  - → (signed binary) Number is -11
  - This method is called "signed-magnitude" and is rarely used in digital systems (if at all)
- In computers, *a negative number is represented by the complement of its absolute value*.
- Signed-complement system
  - positive numbers have always "0" in the MSB position
  - negative numbers have always "1" in the MSB position

Signed-Complement System

- <u>Example</u>:
  - Decimal 11 = $(01011)_2$
  - How to represent –11 in 1's and 2's complements
  1. 1's complement –11 =
  2. 2's complement -11 =
  - If we use eight bit precision:
  - **11 = 00001011**
  - 1's complement **-11 = 11110100**
  - 2's complement **-11 = 11110101**

Signed Number Representation

Signed magnitude		One's complement		Two's complement	
000	+0	000	+0	000	0
001	+1	001	+1	001	+1
010	+2	010	+2	010	+2
011	+3	011	+3	011	+3
100	-0	111	-0	111	-1
101	-1	110	-1	110	-2
110	-2	101	-2	101	-3
111	-3	100	-3	100	-4

- Issues: balance, number of zeros, ease of operations
- Which one is best? Why?

Arithmetic Addition

- Examples:

```
+11        00001011        -11        11110101
 +9    +   00001001         +9    +   00001001
       ------------              ------------
           00010100                   11111110
```

No carry, leftmost bit is 0, result is what you want

```
+11        00001011                          1
 -9    +   11110111         -9    +   11110111
       ------------              ------------
          100000010                  111101100
```

- No special treatment for sign bits

Arithmetic Addition

- Examples:

No carry, leftmost bit is 1, result is negative, take 2s complement, get -2

```
+

          00010100
```

```
-11        11110101
 +9     +  00001001
        -----------
           11111110
```

```
+11        00001011
 -9     +  11110111
        -----------
          100000010
```

```
-11        11110101
 -9     +  11110111
        -----------
          111101100
```

- No special treatment for sign bits

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Arithmetic Addition

- Examples:

```
+11      00001011          -11        11110101
 +9    + 00001001           +9     +  00001001
                                      ---------
                                      11111110
```

Carry=1, leftmost bit is 0, result is what you want

```
+11       00001011         -11        11110101
 -9    +  11110111          -9     +  11110111
          --------                 -----------
         100000010                   111101100
```

- No special treatment for sign bits

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Arithmetic Addition

- Examples:

```
+11        00001011          -11        11110101
 +9      + 00001001           +9      + 00001001
         ----------                   ----------
           00010100                     11111110

+11        0000...           -11        11110101
 -9      + 11110111           -9      + 11110111
         ----------                   ----------
          100000010                    111101100
```

Carry=1, leftmost bit is 1, result is negative, take 2s complement, get -20

- No special treatment for sign bits

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Arithmetic Overflow 1/2

- In hardware, we have limited resources to accommodate numbers
  - Computers use 8-bit, 16-bit, 32-bit, and 64-bit registers for the operands in arithmetic operations.
  - Sometimes the result of an arithmetic operation get too large to fit in a register.

Arithmetic Overflow 2/2

- <u>Example</u>:

```
+2       0010
+4   +   0100
    ---------
         0110

+7       0111
+6   +   0110
    ---------
         1101
```

```
-3       1101
-5   +   1011
    ---------
        10000

-3       1101
-6   +   1010
    ---------
        10111
```

- <u>Rule</u>: If the MSB and the bits to the left of it differ, then there is an overflow

Subtraction with Signed Numbers

- <u>Rule</u>: is the same
- We take the 2's complement of the subtrahend
  - It does not matter if the subtrahend is a negative number.
  - $(\pm A) - (-B) = \pm A + B$

```
 -6      11111010            -6     11111010
-13  -   11110011     →           + 00001101
     ------------               ------------
                                  1 00000111
```

- Signed-complement numbers are added and subtracted in the same way as unsigned numbers
- With the same circuit, we can do both signed and unsigned arithmetic

Alphanumeric Codes

- Besides numbers, we have to represent other types of information
  - letters of alphabet, mathematical symbols.
- For English, alphanumeric character set includes
  - 10 decimal digits
  - 26 letters of the English alphabet (both lowercase and uppercase)
  - several special characters
- We need an alphanumeric code
  - ASCII
  - American Standard Code for Information Exchange
  - Uses 7 bits to encode 128 characters

ASCII Code

- 7 bits of ASCII Code
  - $(b_6\ b_5\ b_4\ b_3\ b_2\ b_1\ b_0)_2$
- Examples:
  - A → **65 = (1000001),…,** Z → **90 = (1011010)**
  - a → **97 = (1100001),** …, z → **122 = (1111010)**
  - **0 →48 = (0110000),…,9 → 57 = (0111001)**
- 128 different characters
  - 26 + 26 + 10 = 62 (letters and decimal digits)
  - 32 special printable characters %, *, $
  - 34 special control characters (non-printable): BS, CR, etc.

Representing ASCII Code

- 7-bit
- Most computers manipulate 8-bit quantity as a single unit (byte)
  - One ASCII character is stored using a byte
  - One unused bit can be used for other purposes such as representing Greek alphabet, italic type font, etc.
- The eighth bit can be used for error-detection
  - parity of seven bits of ASCII code is prefixed as a bit to the ASCII code.
  - A → (**0 1000001**) even parity
  - A → (**1 1000001**) odd parity
  - Detects one, three, and any odd number of bit errors

Binary Logic

- Binary logic is equivalent to what it is called “two-valued Boolean algebra”
  - Or we can say it is an implementation of Boolean algebra
- Deals with variables that take on “two discrete values” and operations that assume logical meaning
- Two discrete values:
  - {true, false}
  - {yes, no}
  - {1, 0}

Binary Variables and Operations

- We use A, B, C, x, y, z, etc. to denote binary variables
  - each can take on {0, 1}
- Logical operations
  1. AND → $x \cdot y = z$ or $xy = z$
  2. OR → $x + y = z$
  3. NOT → $\overline{x} = z$ or $x' = z$
  - For each combination of the values of x and y, there is a value of specified by the definition of the logical operation.
  - This definition may be listed in a compact form called truth table.

Truth Table

x	y	AND x · y	OR x + y	NOT x′
0	0			
0	1			
1	0			
1	1			

Logic Gates

- Binary values are represented as electrical signals
  - Voltage, current
- They take on either of two recognizable values
  - For instance, voltage-operated circuits
  - 0V → 0
  - 4V → 1
- Electronic circuits that operate on one or more input signals to produce output signals
  - AND gate, OR gate, NOT gate

Range of Electrical Signals

- What really matters is the range of the signal value

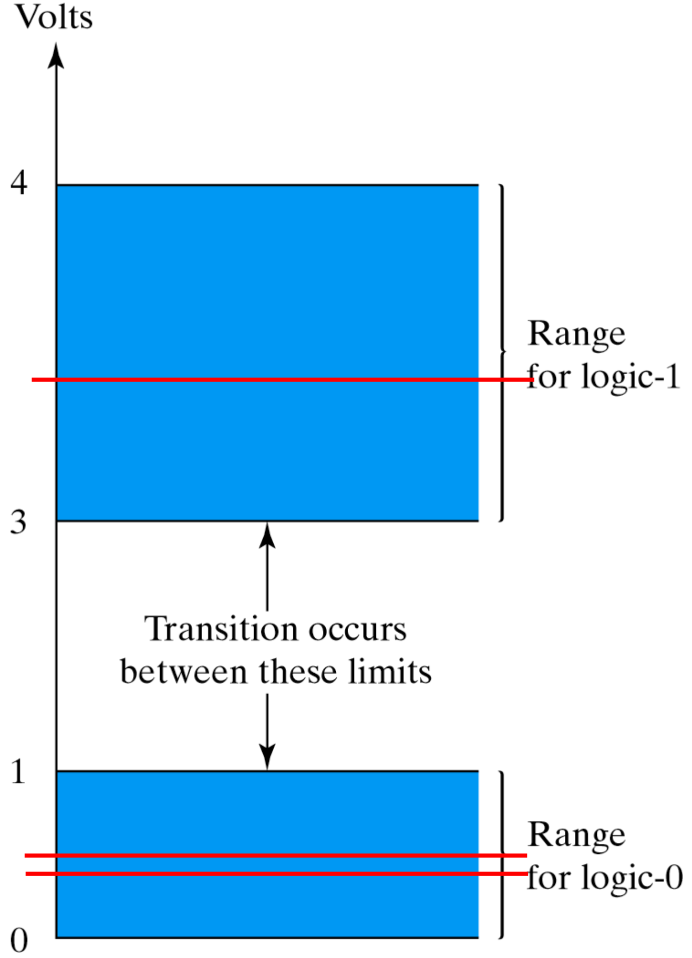
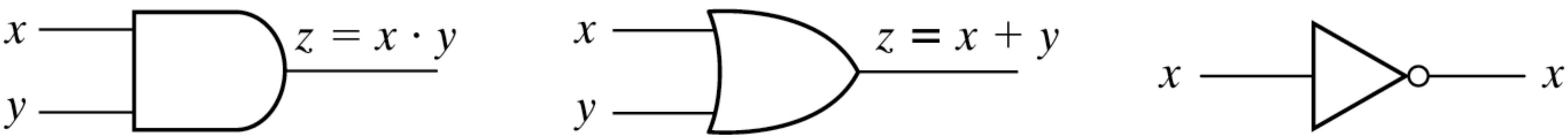


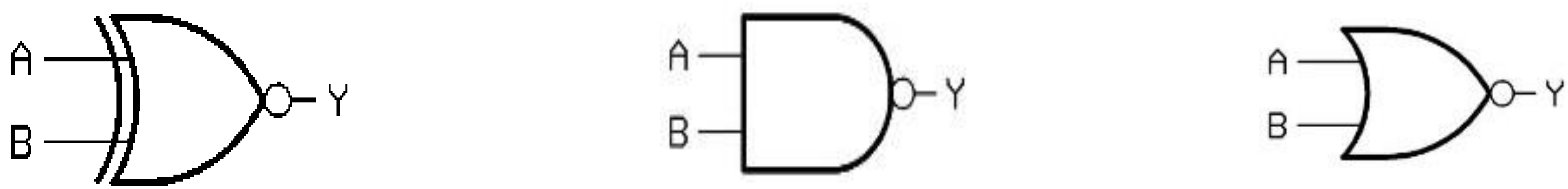
Fig. 1-3 Example of binary signals

Logic Gate Symbols

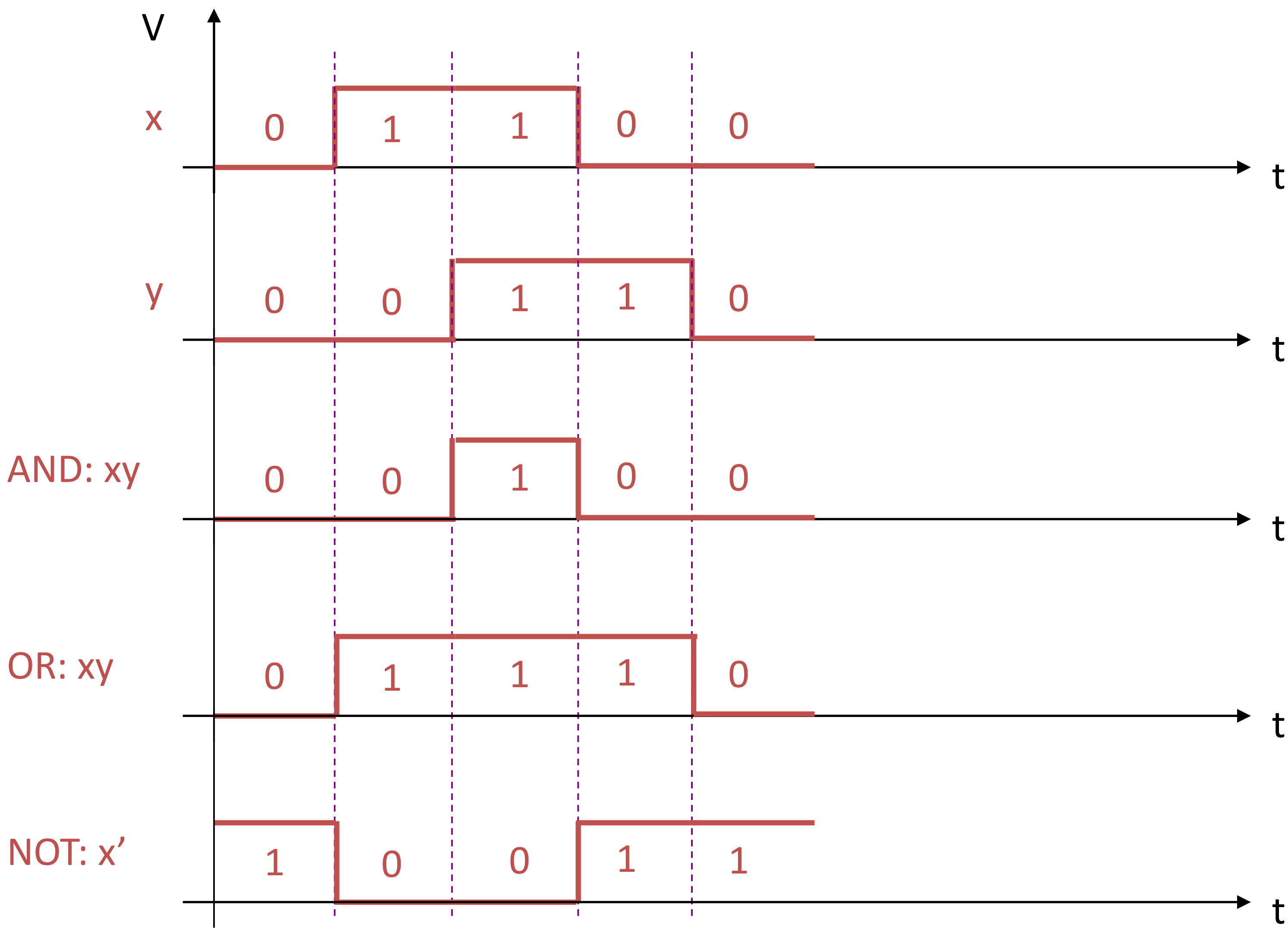


(a) Two-input AND gate (b) Two-input OR gate (c) NOT gate or inverter

Fig. 1-4 Symbols for digital logic circuits

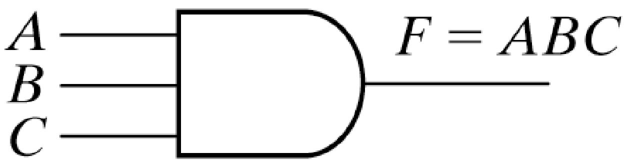


Gates Operating on Signals

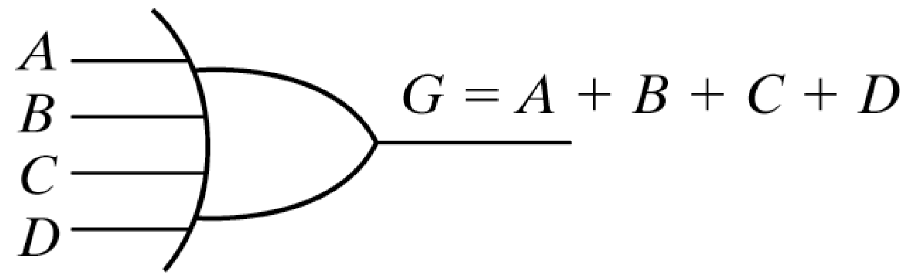


Input-Output Signals for gates

Gates with More Than Two Inputs



(a) Three-input AND gate



(b) Four-input OR gate

Fig. 1-6 Gates with multiple inputs